LAPLACE TRANSFORMS:

Definition: Let $F(t)$ is a function of "t" defined for all positive values of "t" , then the Laplace transform of $f(t)$ is denoted by $L\{F(t)\} = f(p)=f(s)$ and is defined by ,

$$L\left\{F(t)\right\} = f(p)=\int_0^\infty e^{-pt}F(t)dt$$

Provided the integral is convergent. The parameter "p" of "s" may be real or complex number.

Laplace Transform $L\{F(t)\} = f(p)$	**Inverse Laplace Transform** $L^{-1}\{f(p)\}= F(t)$
$L\{1\}=\frac{1}{p}$, $p>0$	$L^{-1}\{\frac{1}{p}\}=1, p>0$
$L\{t\}=\frac{1}{p^2}$, $p>0$	$L^{-1}\{\frac{1}{p^2}\}=t$, $p>0$
$L\{t^n\}=\frac{n!}{p^{n+1}}$, $p>0$	$L^{-1}\{\frac{1}{p^{n+1}}\}=\frac{t^n}{n!}$, $p>0$
$L\{e^{at}\}=\frac{1}{p-a}$, $p>a$	$L^{-1}\{\frac{1}{p-a}\}=e^{at}$, $p>a$
$L\{\cos at\}=\frac{p}{p^2+a^2}$, $p>0$	$L^{-1}\{\frac{p}{p^2+a^2}\}=\cos at$, $p>0$
$L\{\sin at\}=\frac{a}{p^2+a^2}$, $p>0$	$L^{-1}\{\frac{1}{p^2+a^2}\}=\frac{\sin at}{a}$, $p>0$
$L\{\cosh at\}=\frac{p}{p^2-a^2}$, $\|p\|>0$	$L^{-1}\{\frac{p}{p^2-a^2}\}=\cosh at$, $\|p\|>0$
$L\{\sinh at\}=\frac{a}{p^2-a^2}$, $\|p\|>0$	$L^{-1}\{\frac{a}{p^2-a^2}\}=\frac{\sinh at}{a}$, $\|p\|>0$
Linear Property $L\{aF_1(t)+bF_2(t)\} = aL\{F_1(t)\}+bL\{F_2(t)\}$	**:** $L^{-1}\{aF_1(t)+bF_2(t)\} =a L^{-1}F_1(t)+b L^{-1}F_2(t)$
First Shifting (Translation)Theorem $L\{e^{at} F(t)\} = f(p-a)$ where $f(p)= L\{F(t)\}$	***First Shifting (Translation)Theorem*** $L^{-1}\{ f(p-a)\} = e^{at} L^{-1} F(t)\}$,where $f(p)= L\{F(t)\}$
Second Shifting *(Translation)* Theorem : If $G(t)=\begin{cases}F(t-a) & ; t>a\\ 0 & ; t<a\end{cases}$ Then $L\left\{G(t)\right\} = e^{-ap} f(p)$, where $f(p)= L\{F(t)\}$	**Second Shifting *(Translation)* Theorem** $L^{-1}\{e^{-ap} f(p)\}=G(t)=\begin{cases}F(t-a) & ; t>a\\ 0 & ; t<a\end{cases}$, where $f(p)= L\{F(t)$
Change of Scale property $L\{F(at)\}=\frac{1}{a}f(\frac{p}{a})$ where $f(p)= L\{F(t)\}$	**Change of Scale property:** $L^{-1}\{f(ap)\}=\frac{1}{a}F(\frac{t}{a})$, where $F(t)= L^{-1}\{f(p)\}$

Differentiation Theorem $L\{F'(t)\} = p\,L\{F(t)\} - F(0)$ & $L\{F^n(t)\} = p^n L\{F(t)\} - p^{n-1}F(0) - p^{n-2}F'(0) - \ldots - F^{n-1}(0)$	
Integral Theorem If $F(t)$ is piecewise continuous function and $\lvert F(t)\rvert \le Me^{at}$ then $L\{\int_0^t F(x)dx\} = \frac{1}{p}L\{F(t)\}$	
Multiplication Th. $L\{tF(t)\} = (-1)\frac{d}{dp}f(p) = -f'(p)$ $L\left\{t^n F(t)\right\} = (-1)\frac{d^n}{dp^n}f(p)$	***Multiplication Theorem*** $L^{-1}\{p\,f(p)\} = F'(t)$ $L^{-1}\{p^n \frac{d^n}{dp^n}f(p)\} = L^{-1}\{p^n f^n(p)\} = F^n(t)$ *where* $F(t) = L^{-1}\{f(p)\}$
Division Theorem $L\{\frac{F(t)}{t}\} = \int_p^\infty f(p)dp$	***Division Theorem*** $L^{-1}\{\frac{f(p)}{p}\} = F(t) = \int_0^t f(p)dp$ $L^{-1}\{\frac{f(p)}{p^n}\} = F(t) = \int_0^t \ldots\ldots\ldots\int_0^t f(p)dp^n$
Fundamental theorem of periodic function: *If F(t) is a periodic function of period T then* $L\{f(t)\} = \frac{\int_0^T e^{-pt}F(t)dt}{1-e^{-pT}}$	

Q .1. Find the Laplace transform of the Elementary functions:
(i) 1 (ii) t (iii) t^n(iii) $Sinat$ (iv) $cosat$ (v) e^{at} (vi) $sinhat$ (vii) $coshat$

Q .2. Find the Laplace transform of the Elementary functions:
$(i)\ (t^2+1)^2$ **[May2018EC]** $(ii)\ \frac{e^{at}-1}{a}$ **[May2018CE]**$(iii)\ 2\sin t.\cos t$ $[May2018EC]$
$(iv)\ \sin 3t.\sin 4t\ (v) 4\cos^2 t\ (vi) e^{-2t} - e^{-3t}\ (vii)(\sin t - \cos t)^2\ (viii) 3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t$
$(ix) 6\sin 2t - 5\cos 2t$ **[May 2018]**

Q .3. Find the Laplace transforms of(1) $L\left\{\sin\sqrt{t}\right\}$ (2) $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$ [[Ans: (i) $\frac{\sqrt{\pi}}{2p^{3/2}}e^{\frac{-1}{4p}}$ (ii) $\sqrt{\frac{\pi}{p}}e^{\frac{-1}{4p}}$

[Hint: $sinx = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\ \&\ cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots$]

Laplace transform of discontinuous functions:

Q .4. Find the Laplace transforms of $f(t) = \begin{cases} 1\ ;\ 0 \le t < 2 \\ t-1\ ;\ 2 \le t \end{cases}$ **[May 2019**

Q .5. Find the Laplace transforms of $f(t) = \begin{cases} (t-1)^2; & t > 1 \\ 0; & 0 < t < 1 \end{cases}$ **[Ans:** $2\frac{e^{-p}}{s^3}$ **[June 2015, Nov. 18]**

Q .6. Find the Laplace transforms of $f(t) = \begin{cases} \sin t\ ;\ 0 < t < \pi \\ 0\ \ ;\ t > \pi \end{cases}$ [Ans: $\frac{1}{1+p^2}(1+e^{-p\pi})$

Q .7. Find the Laplace transforms of $f(t) = \lvert t-1\rvert + \lvert t+1\rvert$, $t \ge 0$ [Ans: $\frac{2}{p}(1+\frac{e^{-p}}{p})$

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Q .8. [Hint : $f(t)=\begin{cases}-(t-1)+(t+1); & 0\le t\le 1\\ (t-1)+(t+1); & t>1\end{cases}$ $\therefore$

$|t-1|=-(t-1), t<1$ & $|t-1|=(t-1), t>1$ *and* $|t+1|=t+1, t>0$]

Q .9. Write three properties of Laplace Transform. **[May 2019]**

FIRST SHIFTING THEOREM ***(Translation)Theorem*** $L\{e^{at}F(t)\}=f(p-a)$ *where* $f(p)=L\{F(t)\}$

Q .10. State and prove first shifting theorem. **[June 2014]**

Q .11. Find the Laplace transforms of (1) $L\{e^{-4t}\sin 3t\}$ **[Dec.2007]** (2) $L\{e^{-t}(3\sinh 2t-5\cosh 2t)\}$

(3) $L\{e^{-3t}t^n\}$ (4) $L\{e^t\sin^2 t\}$ **[May 2018]** [Ans: (i) $\frac{3}{(p+4)^2+3^2}$ (ii) $\frac{1-5p}{p^2+2p-3}$ (iii) $\frac{n!}{(p+3)^{n+1}}$

Q .12. State and prove Second shifting theorem.

Or If $L\{F(t)\}=f(p)$ and $G(t)=\begin{cases}F(t-a), & t>a\\ 0, & t<a\end{cases}$ then $L\{G(t)\}=e^{-ap}f(p)$

Q .13. State and prove Change of Scale property. { If $L\{F(t)\}=f(p)$ then $L\{F(at)\}=1/a\ f(p/a)$

MULTIPLICATION THEOREM [$L\{t^nF(t)\}=(-1)^n\frac{d^n}{dp^n}f(p)$] *where* $f(p)=L\{F(t)\}$

Q .14. Find (1) $L\{t\sin at\}$ **[May 2018 EC]** (2) $L\{t^2\sin at\}$ **[Dec. 2004 , 2010, 2011,May 2018]**

(3) $L\{t^4e^{-3t}\}$ **[June 2016]** (4) $L\{t^2\cos at\}$ (5) $L\{te^{-t}\sin at\}$ **[RGPV. June 2006,2016]**

(6) $L\{t^ne^{-at}\}$ [**May 25018 EC**] (7) $L\{te^{-4t}\sin 3t\}$ **[May2018]ME**

Q .15. Find (1)L{ $t^2e^{-2t}\cos 3t$} **[June 2014]** (2) $L\{te^{-t}\sin at\}$ (3) $L\{te^{-t}\sin 3t\}$ [June 2006]

[Ans: (1) (2) $\frac{2a(p+1)}{\{(p+1)^2+a^2\}^2}$ (3) $\frac{6(p+1)}{(p^2+2p+10)^2}$

DIVISION THEOREM: $L\{\frac{F(t)}{t}\}=\int_p^\infty f(p)dp$ *where* $f(p)=L\{F(t)\}$

Q .16. Find the Laplace transforms of $L\left\{\frac{\sin t}{t}\right\}$ and obtain $L\left\{\frac{\sin at}{t}\right\}$ [**Ans: $\tan^{-1}$ (1/p) & $\tan^{-1}$ (a/p)**

Q .17. Find (i) $L\left\{\frac{1-\cos 2t}{t}\right\}$[**Ans:** $\frac{1}{2}\log(\frac{p^2+4}{p^2})$][***Dec. 2003, June 2007, 2012]***

(ii) $L\left\{\frac{1-e^t}{t}\right\}$ **[Ans:** $\log(\frac{p-1}{p})$ **Dec. 2011]** **(iii)** $L\left\{\frac{e^{at}-e^{bt}}{t}\right\}$ **[May 2018]** **[Ans:** $\log(\frac{p+b}{p+a})$ **]**

Q .18. Show that $L\left\{\frac{\cos at}{t}\right\}$ does not exist but $L\left\{\frac{\cos at-\cos bt}{t}\right\}$ exist and find it. **[Dec. 2010June 2012]**

Q .19. Find the Laplace transform of $L\left\{\frac{e^{-t}\sin t}{t}\right\}$ [**Ans: $\cot^{-1}$(s+1)]**

LAPLACE TRANSFORM OF DERIVATIVES:

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Q .20. Prove that : If F(t) is continuous for all t>0 and be of exponential order "*a*" as $t\to\infty$ and if *F'(t)* is of class *A* , then Laplace transform of *F'(t)* exist and $L\{F'(t)\}=pL\{F(t)\}-F(0)$ **[June 2001]**

LAPLACE TRANSFORM OF INTEGRALS: [$L\{\int_0^t F(x)dx\}=\frac{1}{p}L\{F(t)\}$

Q .21. Find (1) $L\left\{\int_0^t \frac{\sin t}{t}dt\right\}$ (2) $L\left\{\int_0^t \frac{e^t \sin t}{t}dt\right\}$ [***Sep. 2009, June 14]*** [**[Ans:** (1) $\cot^{-1}p$ (2) $1/p\ \cot^{-1}(p-1)$

EVALUATION OF INTEGRALS USING LAPLACE TRANSFORM:

Q .22. Evaluate (1) $\int_0^\infty \frac{e^{-t}\sin t}{t}dt$ (2) $\int_0^t \frac{\sin at}{t}dt$ (3) $\int_0^\infty \frac{\cos at-\cos bt}{t}dt$ 4) $\int_0^\infty te^{-3t}\sin t\,dt$ (5) $\int_0^\infty t^3 e^{-t}\sin t\,dt$

[Ans: (1) (2) tan-1 (a/p) (3) $\frac{1}{2}log\frac{p^2+b^2}{p^2+a^2}$ (4)3/50 (5) 0

Q .23. Using Laplace transform Prove that (1) $\int_0^\infty \frac{\sin t}{t}dt=\frac{\pi}{2}$ (2) $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}dt=\log\frac{b}{a}$

(3) $\int_0^\infty t\,e^{-2t}\cos t\,dt=\frac{3}{25}$

Laplace Transform of Some Special Functions :

Q .24. Find Laplace transform of (i) **Sine Integral Function** $Si(t)=\int_0^t \frac{\sin u}{u}du$ [Ans: ***1/p cot***$^{-1}$***p***]

(iii) **Unit Step Function or Heaviside's function** $H(t-a)=\begin{cases}0, & t<a\\ 1, & t>a\end{cases}$

[Hint: use discontinuous function formula L{H(t-a)}= e^{-ap}/p]

(iv) **Dirac Delta Function or Unit Impulse function** $F\varepsilon(t)=\begin{cases}\frac{1}{\varepsilon}, & 0\le t\le\varepsilon\\ 0, & t>\varepsilon\end{cases}$ **[Dec. 2003, Dec. 2006]**

[Ans: $1/s\varepsilon\ (1-e^{-s\varepsilon})$]

Q .25. Let *F(t) = f(p)* be a periodic function with period *T* , then prove that $L\{F(t)\}=\frac{\int_0^T e^{-pt}F(t)dt}{1-e^{-pT}}$ ***[RGPV Jan. 2007,Dec. 2011]***

Q .26. If L *{F(t)} = f(p)* then prove that *L {t F(t)}=-f'(p)* ***[RGPV. Dec. 2001]***

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INVERSE LAPLACE TRANSFORMS:

First Shifting (Translation) Theorem : $L^{-1}\{f(p-a)\} = e^{at} L^{-1} F(t)\}$, where $f(p) = L\{F(t)\}$

Q.27 Find the Inverse Laplace Transform of (i) $L^{-1}\left\{\frac{6}{2p-3} - \frac{3+4p}{9p^2-16} + \frac{8-6p}{16p^2+9}\right\}$ (ii) $L^{-1}\left\{\frac{1}{p^2-6p+18}\right\}$

(iii) $L^{-1}\left\{\frac{3p+7}{p^2-2p-3}\right\}$ [**May 2018 EC**] (iv) $L^{-1}\left\{\frac{1}{9p^2+2p}\right\}$ **[June 2016]**

(v) $L^{-1}\left\{\frac{3p-2}{p^2-4p+20}\right\}$ **[May 2018 CE,EC]** (vi) $L^{-1}\left\{\frac{5p+3}{(p-1)(p^2+2p+5)}\right\}$ **[May 2018] ME**

Q .17. Find (1) $L^{-1}\left\{\frac{1}{p^2-6p+10}\right\}$ **[May 2018]** (2) $L^{-1}\left\{\frac{p+1}{(p+2)^2}\right\}$

(3) $L^{-1}\left\{\frac{3}{(p+1)^2}\right\}$ (4) $L^{-1}\left\{\frac{6p^2-15p-11}{(p+1)(p-2)^3}\right\}$ **[Nov18]**

(3) $L^{-1}\left\{\frac{2p^2-6p+5}{(p-1)(p-2)(p-3)}\right\}$ (5) $L^{-1}\left\{\frac{6p^2+22p+18}{p^3+6p^2+11p+6}\right\}$ **[June 2015]**

Convolution Theorem: If $L^{-1}\{f(p)\} = F(t)$ and $L^{-1}\{g(p)\} = G(t)$, where F and G are two function of Class A then $L^{-1}\{f(p).g(p)\} = \int_0^t F(x)G(t-x)dx = F*G$

Q .18. Use **Convolution theorem** to evaluate. (1) $L^{-1}\left\{\frac{1}{(p+1)(p-2)}\right\}$ **[May2018CE, EC]**

(2) $L^{-1}\left\{\frac{1}{(p^2+a^2)^2}\right\}$ **(3)** $L^{-1}\left[\frac{1}{(p+1)(p^2+1)}\right]$ **[June 2014]**

(4) $L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\}$ **[RGPV.June 2008, June, Dec. 2011, 2012, Dec, 2011, June 2015, June 2016]**

(5) $L^{-1}\left\{\frac{1}{(p+3)(p^2+9)}\right\}$ **[June 2011 ,Nov.18]** (6) $L^{-1}\left\{\frac{p^2}{(p^2+a^2)(p^2+b^2)}\right\}$ **[June 2006, 2008,Dec. 2008,2010]**

Heaviside's Expansion Theorem: If *f(p)* and *g(p)* are two polynomials in *p* ,where *degree f(p) <degree g(p)* . If g(p) is a polynomial of n- distinct zeros $\alpha_1, \alpha_2, \ldots\ldots\ldots\alpha_n$ then

$$L^{-1}\left\{\frac{f(p)}{g(p)}\right\} = \sum_{i=1}^{n} \frac{f(\alpha_i)}{g'(\alpha_i)} e^{\alpha_i t} = \frac{f(\alpha_1)}{g'(\alpha_1)} e^{\alpha_1 t} + \frac{f(\alpha_2)}{g'(\alpha_2)} e^{\alpha_2 t} + \ldots\ldots\ldots \frac{f(\alpha_i)}{g'(\alpha_i)} e^{\alpha_i t}$$

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Evaluate $L^{-1}\left\{\frac{p^2+6}{(p^2+1)(p^2+4)}\right\}$ **[RGPV. June 2002, 2012 Jan. 2006]**

Q .23. Evaluate $L^{-1}\left\{\frac{p+2}{(p^2+4p+5)^2}\right\}$ **[RGPV. Sep. 2009]**

Inverse Laplace transform using multiplication theorem:

$$L\{tF(t)\}=(-1)\frac{d}{dp}f(p)=-f'(p) \quad or \quad L\left\{t^nF(t)\right\}=(-1)\frac{d^n}{dp^n}f(p)$$

Q .24. Find (i) $L^{-1}\left\{\log\frac{p(p+1)}{(p^2+4)}\right\}$ ***[RGPV. Dec. 2004]*** (2) $L^{-1}\left\{\log\frac{p+1}{p-1}\right\}$ **[June 2005, Feb. 2010]**

(3) $L^{-1}\left\{\log\frac{p^2-1}{p^2}\right\}$ **[Jan2006]** (4) $L^{-1}\left[\log\frac{p+1}{P+3}\right]$ **[June 2014]**

(5) $L^{-1}\left\{\log(1+\frac{1}{p})\right\}$ (6) $L^{-1}\{\tan^{-1}(p/2)\}$ (ii) $L^{-1}\{\tan^{-1}(a/p)\}$ **[Dec. 2012]**

Differentiation Theorem $L\{F'(t)\}=p\,L\{F(t)\}-F(0)$

And $L\{F^n(t)\}=p^n\,L\{F(t)\}-p^{n-1}F(0)-p^{n-2}F'(0)-\ ....\ -F^{n-1}(0)$

SOLUTION OF DIFFERENTIAL EQUATIONS BY LAPLACE TRANSFORMS

Q .25. Using Laplace Transformation solve the following differential equation $\frac{d^2y}{dt^2}+9y=6\cos 3t$ y(0)=2, y'(0)=0,

Q .26. Using L. T. solve the following differential equation $\frac{d^2x}{dt^2}+9x=\cos 2t$,if $x(0)=1$, $x(\frac{\pi}{2})=-1$

[RGPV .June. 2008]

Q .27. Using Laplace Transformation solve the following differential equation
$y''-2y'+y=e^t$, $y(0)=2$, $y'(0)=-1$ ***[RGPV. Dec. 2008 , 2011 , Feb. 2010,Nov.18]***

Q .28. Using Laplace Transformation solve the following differential equation
$y''-3y'+2y=4t+e^{3t}$, $y(0)=1$, $y'(0)=-1$ ***[RGPV. May 2018]ME***

Q .29. Using Laplace Transformation solve the following differential equation $\frac{d^3y}{dt^3}-3\frac{d^2y}{dt^2}+3\frac{dy}{dt}-y=t^2e^t$

Where $y(0)=1$, $y'(0)=0$, $y''(0)=-2$ ***[RGPV. Dec. 2007]***

Q .30. Using Laplace Transformation solve the following differential equation $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y=\sin t$

Where $y(0)=1$, $y'(0)=0$, ***[RGPV. Dec. 2007]***

Q .31. Using Laplace Transformation solve the following differential equation $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y=3e^{-x}\sin x$

Where $y(0)=0, y'(0)=1$, ***[RGPV. June. 2007]***

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Q .32. Using Laplace Transformation solve the following differential equation $\frac{d^3y}{dx^3}+2\frac{d^2y}{d^2x}-\frac{dy}{dx}-2y=0$

Where *y(0) =1, y'(0) =2, y''(0) =2*

Q .33. Solve the following diff. equation using Laplace transform $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=5e^t$, *y(0) =2* , *y'(0)=1* **[June14]**

Simultaneous differential equations:

Q .34. Solve the following simultaneous differential equation by Laplace transform

$3\frac{dx}{dt}-y=2t$, $\frac{dx}{dt}+\frac{dy}{dt}-y=0$ With the conditions *x(0) = y(0) = 0*

Q .35. Solve the following simultaneous differential equation by Laplace transform $\frac{dx}{dt}+y=\sin t$,

$\frac{dy}{dt}+x=\cos t$ With the conditions *x(0)=2 , y(0) = 0*

Q .36. Solve the following simultaneous differential equation by Laplace transform

$\frac{dx}{dt}+5x-2y=t$, $\frac{dy}{dt}+2x+y=0$ With the conditions *x(0) = y(0) = 0* ***[RGPV. Sep. 2009]***

Fourier Transform

If *f(x)* is the function defined in the interval (-∝ , ∝) , uniformly continuous in the finite intervals and $\int_{-\infty}^{\infty}|f(x)|dx$ converges then the Fourier transform of a one-dimensional function f(x) is defined as

$$\Im[f(x)]=F(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x)e^{isx}\,dx$$ [Note: One can leave coefficient 1/2π] .

The inverse transform $\Im^{-1}$ is defined as

$$\Im^{-1}[F(s)]=f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(s)e^{-isx}\,ds$$, Where "*s*" is a parameter.

It may be represented by

$$\Im[f(x)]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{isx}\,dx$$

and

$$\Im^{-1}[F(s)]=f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}\,ds$$

Remark: 1. Since $|x|=\begin{cases}x, & x>0\\ -x, & x<0\end{cases}$ hence means $x>a$ and $-x<a$ ⇒ $x>a$ and $x>-a$ or $-a<x<a$

2. $|x|>a$ means $-\infty.........-a..........0.............a............\infty$: a< x < ∝ and -∝ < x < -a

Thus $f(x)=\begin{cases}x, & |x|\le a\\ 0, & |x|\ge a\end{cases}=\begin{cases}x, & -a<x<a\\ 0, & -\infty<x<-a \text{ and } a<x<\infty\end{cases}$

➢ ***For evaluation of integrals we use inverse Fourier transform.***

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Fourier Transform (or Fourier Complex Transform):

Q. 1 . Write Linear and Change of scale property for Fourier Transform. **[June 2014]**

Q. 2 . State and prove shifting property for Fourier Transform. **[Hint:** $F\{f(x-a)\}=e^{isa}f(s)$]

Q. 3 . State and Prove **Convolution Theorem** for Fourier Transform.

Q. 4 . Find the Fourier complex transform of *f(x)* ,if $f(x) = \begin{cases} e^{iwx}, & a < x < b \\ 0, & x < a,\ x > b \end{cases}$

[Ans: $F\{f(x)\} = -\frac{i}{s+w}[e^{i(s+w)b} - e^{i(s+w)a}]$]

Q. 5 . Find the Fourier complex transform of *f(x)* ,if $f(x)=\begin{cases} \frac{\sqrt{2\pi}}{2\varepsilon}, & |x|\le\varepsilon \\ 0, & |x|>\varepsilon \end{cases}$ **[Ans:** $F\{f(x)\}=\frac{\sin s\varepsilon}{s\varepsilon}$ **]**

Q. 6 . Find the **Fourier transform** of $f(x) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$ **[May 2018, 2019]**

Q. 7 . Find the Fourier transform of $f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$ **[Nov. 2018]**

Hence evaluate

(i) $\int_0^\infty \frac{\sin sa \cos sx}{s} ds$ (ii) $\int_0^\infty \frac{\sin s}{s} ds$ (iii) $\int_0^\infty \frac{\sin x}{x} dx$ **[2003, June 17] [Ans: (i)** $\frac{\pi}{2}$ **(ii)** $\frac{\pi}{2}$ **(iii)** $\frac{\pi}{2}$ **]**

Q. 8 . Find the **Fourier transform** of $f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$ Hence evaluate $\int_0^\infty \frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}dx$

[Ans: -3π/16]

Q. 9 . Find Fourier Transform of **Dirac Delta Function**.

[Hint : Dirac Delta function $\delta(t-a) = \lim_{h\to 0} I(h,t-a) = \lim_{h\to 0}\begin{cases} \frac{1}{h}, & a<t<a+h \\ 0, & t<a, t>a+h \end{cases}$ Ans: $\frac{e^{isa}}{\sqrt{2\pi}}$]

Q. 10 . Find the Fourier **transform** of the function , $f(x)=e^{-ax^2}$,$a>0$ **Ans:** $\sqrt{\frac{\pi}{a}}e^{-\frac{s^2}{4}}$ **] [June 2015, 17]**

Q. 11 . Find the Fourier **transform** of the function , $f(x)=e^{-x^2}$ **[Ans:** $\sqrt{\pi}e^{-\frac{s^2}{4}}$ **]**

Q. 12 . Show that the **Fourier transforms** of $f(x)=e^{-\frac{x^2}{2}}$ is self reciprocal.

Q. 7 . Find Fourier Transform of $f(x)=e^{-|x|}$ (or . $f(x)=e^{-a|x|}$). [Ans : $f(x)=\sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}$]

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Fourier Sine Transform:

$$\Im_s\left[f(x)\right] = F_s(s) = \int_0^\infty f(x)\sin sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

and its inverse transform $\Im^{-1}{}_s\left[F(s)\right] = f_s(x) = \int_0^\infty F(s)\sin sx\,ds = \sqrt{\frac{2}{\pi}}\int_0^\infty F(s)\sin sx\,ds$

Fourier Cosine Transform:

$$\Im_c\left[f(x)\right] = F_c(s) = \int_0^\infty f(x)\cos sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(x)\cos sx\,dx$$

And its inverse transform

$$\Im^{-1}{}_c\left[F(s)\right] = f_c(x) = \int_0^\infty F(s)\cos sx\,ds = \sqrt{\frac{2}{\pi}}\int_0^\infty F(s)\cos sx\,ds = \frac{1}{\sqrt{2\pi}}\int_0^\infty F(s)\cos sx\,ds$$

Fourier Sine and Cosine Transform:

Q. 8 . Find the Fourier sine transform of $f(x)= e^{-3x}+e^{-4x}$ [**Ans:** $\frac{s}{9+s^2}+\frac{s}{16+s^2}$] **[June 17]**

Q. 9 . Find the **cosine transform** of the function $f(x)=\begin{cases}\cos x & ;\ 0<x<a\\ 0 & ;\ x>a\end{cases}$ **[Ans:** $\frac{\sin(1+s)a}{1+s}+\frac{\sin(1-s)a}{1-s}$

Q. 10 . Find the **sine transform** of the function , $f(x)=\begin{cases}\sin x & ;\ 0<x<a\\ 0 & ;\ x>a\end{cases}$ **Ans:** $\frac{\sin(1-s)a}{1-s}-\frac{\sin(1+s)a}{1+s}$

[Dec. 2014 Nov. 2018]

Q. 11 . Find Fourier sine and cosine transform of e^{-x} and recover the original function using inverse formula.

Q. 12 . Using Fourier integral show that $\int_0^\infty \frac{\lambda\sin\lambda x}{\lambda^2+k^2}d\lambda = \frac{\pi}{2}e^{-kx}$, $k>0, x>0$ **[Hint:** Find F.Sine T. of $f(x)= e^{-kx}$**]**

Q. 13 . Find the **sine and cosine transform** of the function , $f(x)=e^{-ax}$ **[May 2018] ME**

Q. 14 . Find the cosine transform of the function $f(x)=e^{-|x|}, x\ge 0$ and prove that $\int_0^\infty \frac{\cos sx}{1+s^2}ds = \frac{\pi}{2}e^{-x}$

Q. 15 . Find the Fourier sine transform of the function $f(x)=e^{-|x|}, x\ge 0$ and prove that

$$\int_0^\infty \frac{x\sin mx}{1+x^2}dx = \frac{\pi}{2}e^{-m}, m>0$$

[June2014]

Q. 16 . Find the Fourier Sine transform of $f(x) = \begin{cases} x & ;0<x<1\\ 2-x & ;1<x<2\\ 0 & ;x>2\end{cases}$ **[June 17]**

Q. 17 . Find Fourier Sine and Cosine Transform of $f(x)=e^{-|x|}$. Hence evaluate $\int_0^\infty \frac{x\sin mx}{1+x^2}dx$ **.[Dec. 2011]**

Q. 18 . Prove that (i) $F_s\{xf(x)\} = -\frac{d}{ds}F_c\{f(x)\}$ (ii) $F_c\{xf(x)\} = \frac{d}{ds}F_s\{f(x)\}$. Hence evaluate Fourier cosine and sine transform of $f(x) = xe^{-ax}$.

Fourier transforms using Differentiation:

Q. 19 . Find the **sine transform** of the function , $f(x) = \frac{1}{x}$

Q. 20 . Find the **Fourier sine transform** of $f(x) = \frac{e^{-ax}}{x}$. Hence find the **Fourier sine transform of** $1/x$.
[June 2016,17]May 2018 CE, [May 2018 EC]

Q. 21 . Find **Fourier cosine transform** of $f(x) = \frac{1}{1+x^2}$ **[May18]ME**

Q. 22 . Find the Fourier sine(and Cosine) transform of $f(x) = e^{-x^2}$. Hence find the Fourier sine transform of *1/x*. [**Hint** Find F. Sine / Cosine T. and Diff. the eq. to make first order diff eq. and solve it]

USEFUL FORMULAE

FACTORIZATION OF THE SUM OR DIFFRENCE OF TWO ANGLES FORMULAE

$(i)\ 2\sin A\cos B = \sin(A+B) + \sin(A-B)$, $(ii)\ 2\cos A\sin B = \sin(A+B) - \sin(A-B)$
$(iii)\ 2\cos A\cos B = \cos(A+B) + \cos(A-B)$, $(iv)\ 2\sin A\sin B = \cos(A-B) - \cos(A+B)$

(*MULTIPLE ANGLE*) *FORMULAE*

$(i)\ \sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$, $(ii)\ \tan 2A = \frac{2\tan A}{1-\tan^2 A}$

$(iii)\ \cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$

$(iv)\ \sin 3A = 3\sin A - 4\sin^3 A$, $(v)\ \cos 3A = 4\cos^3 A - 3\cos A$, $(vi)\ \tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

HALF ANGLE FORMULA

$(i)\ \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$, $(ii)\ \tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

$(iii)\ \cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = 1 - 2\sin^2\frac{A}{2} = 2\cos^2\frac{A}{2} - 1 = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

$(iv)\ 1 - \cos A = 2\sin^2(A/2)$, $1 + \cos A = 2\cos^2(A/2)$

HYPERBOLIC FUNCTIONS

$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$ $\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$ $\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1 - e^{-2x}}{1 + e^{-2x}}$

$\operatorname{sech} x = \frac{1}{\cosh x}$, $\operatorname{cosech} x = \frac{1}{\sinh x}$, $\tanh x = \frac{1}{\coth x} = \frac{\sinh x}{\cosh x}$

$\cosh(-x) = \cosh x$ $\tanh(-x) = -\tanh x$

Log forms of hyperbolic functions :

$\cosh^{-1} x = \ln\left\{x + \sqrt{x^2 - 1}\right\}, \quad x \geq 1$	$\sinh^{-1} x = \ln\left\{x + \sqrt{x^2 + 1}\right\}$, all x	$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), -1 < x < 1$

Properties of Hyperbolic Functions:

$\cosh^2 x - \sinh^2 x = 1$	$1 - \tanh^2 A = \operatorname{sech}^2 A$	$2\sinh^2 x + 1 = \cosh 2x$
$\sinh 2x = 2\cosh x \sinh x$	$\cosh 2x = \cosh^2 x + \sinh^2 x$	$2\cosh^2 x - 1 = \cosh 2x$
$\sinh(A+B) = \sinh A \cosh B + \cosh A \sinh B$	$\cosh(A+B) = \cosh A \cosh B + \sinh A \sinh B$	

Some Useful formulas: LIMIT OF SOME SPECIAL FUNCTIONS

(i) $\lim_{x\to\infty} \frac{1}{x} = 0$ (ii) $\lim_{x\to-\infty} (1+\frac{1}{x})^{x} = e$ (iii) $\lim_{x\to0}(1+x)^{\frac{1}{x}} = e$

(iv) $\lim_{x\to0} \frac{\sin x}{x} = 1 = \lim_{x\to0} \frac{\tan^{-1} x}{x} = \lim_{x\to0} \frac{\sin^{-1} x}{x} = \lim_{x\to0} \frac{\tan^{-1} x}{x}$

(v) $\lim_{x\to\infty} \frac{e^{x} - 1}{x} = 1$ (vi) $\lim_{x\to\infty} \frac{a^{x} - 1}{x} = \ln a \ ,a > 0$ (v) $\lim_{x\to\infty} \frac{x^{n} - a^{n}}{x - a} = na^{n-1}$

INDETERMINATE FORMS $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, 0^{0}, \infty^{0}, \infty-\infty, 1^{\infty}$ resolve indeterminate form before using the limit by using L-hospital rule or by solving the fractions.

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DIFFERENTIAL AND INTEGRAL CALCULUS

First Principle: The *derivative* of the function *f(x)* is the function *f'(x)* defined by

$$f'(x) \equiv \frac{d}{dx}\left[f(x)\right] \equiv \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

S.No	Differentiation	Integration
1	$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1}+c,\quad n\neq -1$
2	$\frac{d}{dx}e^{ax} = ae^{ax}$	$\int e^{ax}dx = \frac{e^{ax}}{a}$
3	$\frac{d}{dx}\log_e x = \frac{1}{x}$	$\int \frac{1}{x}dx = \log x$
4	$\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e$	$\int a^x dx = \frac{a^x}{\log_e a}$
5	$\frac{d}{dx}\sin ax = a\cos ax$	$\int \sin ax\, dx = -\frac{\cos ax}{a}$
6	$\frac{d}{dx}\cos ax = -a\sin ax$	$\int \cos ax\, dx = \frac{\sin ax}{a}$
7	$\frac{d}{dx}\tan ax = a\sec^2 ax$	$\int \tan ax\, dx = \frac{-\log\sec ax}{a} = \frac{\log\cos ax}{a}$ $\int \sec^2 ax\, dx = \frac{\tan ax}{a}$
8	$\frac{d}{dx}\cot ax = -a\cos ec^2 ax$	$\int \cot ax\, dx = \frac{-\log\cos ec ax}{a} = \frac{\log\sin ax}{a}$ $\int \cos ec^2 ax\, dx = \frac{-\cot ax}{a}$
9	$\frac{d}{dx}\sec ax\, dx = a\sec ax\tan ax$	$\int \sec ax\tan ax\, dx = \frac{\sec ax}{a}$ $\int \sec x\, dx = \log(\sec x+\tan x) = \log\ \tan(\frac{\pi}{4}+\frac{x}{2})$
10	$\frac{d}{dx}\cos ec ax = -a\cos ec ax\cot ax$	$\int \cos ec ax.\cot ax\ dx = \frac{-\cot ax}{a}$ $\int \cos ec\, x\, dx = \log(\cos ec\, x-\cot x) = \log\tan\frac{x}{2}$
11	$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x$
12	$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx = -\cos^{-1}x$
13	$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}dx = \tan^{-1}x$
14	$\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$	$\int \frac{1}{1+x^2}dx = -\cot^{-1}x$

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15	$\frac{d}{dx}\sec^{-1}x=\frac{1}{x\sqrt{x^2-1}}$	$\int\frac{1}{x\sqrt{x^2-1}}dx=\sec^{-1}x$
16	$\frac{d}{dx}\cos ec^{-1}x=-\frac{1}{x\sqrt{x^2-1}}$	$\int\frac{1}{x\sqrt{x^2-1}}dx=-\cos ec^{-1}x$
17	**MULTIPLICATION FORMULA** $\frac{d}{dx}f_1(x).f_2(x)=f_2(x).\frac{d}{dx}f_1(x)+f_1(x).\frac{d}{dx}f_2(x)$	**MULTIPLICATION FORMULA** $\int u.v\,dx=u\int v\,dx-\int\{\frac{d}{dx}u.\int vdx\}dx$
18	**DIVISION FORMULA(Quotient Rule)** $\frac{d}{dx}(\frac{f_1}{f_2})=\frac{f_2.(\frac{d}{dx}f_1)-f_1.(\frac{d}{dx}f_2)}{(f_2)^2}$	**Leibnitz′successive integration by Parts** $=u\int vdx-u'\int\int v\,dx^2+u''\int\int\int vdx^3..............\underbrace{\int\int\int}_{n} vdx^n$
19	$\frac{d}{dx}\sqrt{x}=\frac{1}{2\sqrt{x}}$	$\int\frac{1}{\sqrt{x}}dx=\int x^{-1/2}dx=\frac{x^{1/2}}{1/2}$

Some Other Formulae for Integration

$\int\frac{1}{\sqrt{a^2-x^2}}dx=\frac{1}{a}\sin^{-1}\frac{x}{a}$	$\int\frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\frac{x}{a}$
$\int\frac{1}{a^2-x^2}dx=\frac{1}{2a}\log(\frac{a+x}{a-x})=\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)$, $-a<x<a$	
$\int\frac{1}{\sqrt{a^2+x^2}}dx=\log(x+\sqrt{a^2+x^2})=\sinh^{-1}\left(\frac{x}{a}\right)$	$\int\frac{1}{\sqrt{x^2-a^2}}dx=\log(x+\sqrt{x^2-a^2})=\cosh^{-1}\left(\frac{x}{a}\right)$
$\int\sqrt{a^2-x^2}dx=\frac{1}{2}[x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}]$	
$\int\sqrt{x^2+a^2}dx=\frac{1}{2}[x\sqrt{x^2+a^2}+a^2\log(x+\sqrt{x^2+a^2})]$	$\int\sqrt{x^2-a^2}dx=\frac{1}{2}[x\sqrt{x^2-a^2}+a^2\log(x-\sqrt{x^2+a^2})]$
$\int e^{ax}.\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}[a\sin bx-b\cos bx]$	$\int e^{ax}.\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}[a\cos bx+b\sin bx]$

Differentiation and Integration of Hyperbolic Functions:

$f(x)$	$\sinh x$	$\cosh x$	$\tanh x$	$\text{sech}\,x$	$\text{cosech}\,x$	$\coth x$
$\frac{d}{dx}f(x)$	$\cosh x$	$\sinh x$	$\sec^2 h\,x$	$-\tanh x\,\text{sech}\,x$	$-\text{cosech}\,x\coth x$	$\text{cosech}^2\,x$
$\int f(x)dx$	$\cosh x$	$\sinh x$	$\log\cos hx$	$\tan^{-1}(\sin hx)$	$\log\tan x/2$	$\log\sin hx$

Definite Integral:

1. $\int_a^b f(x)\text{dx}=\int_a^b f(y)\text{dy}=\int_a^b f(t)\text{dt}$.
2. $\int_a^b f(x)\text{dx}=-\int_b^a f(x)\text{dx}$
3. $\int_a^a f(x)\text{dx}=\int_b^b f(x)\text{dx}=\int_a^b 0\,\text{dx}=0$
4. Let $a\le c\le b$, then $\int_a^b f(x)\text{dx}=\int_a^c f(x)\text{dx}+\int_c^b f(x)\text{dx}$

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5. (i) If $f(-x) = f(x)$ **(Even Function)** then $\int_{-a}^{a} f(x)\,\mathrm{dx} = 2\int_{0}^{a} f(x)\,\mathrm{dx}$

   (ii) If $f(-x) = -f(x)$ **(Odd Function)** then $\int_{-a}^{a} f(x)\,\mathrm{dx} = 0$

6. If $f(x)$ is periodic function, with period T i.e. $f(x+T) = f(x)$

   (a) $\int_{\alpha}^{\beta} f(x)\mathrm{dx} = \int_{\alpha+T}^{\beta+T} f(x)\mathrm{dx}$ (b) $\int_{0}^{\alpha} f(x)\mathrm{dx} = \int_{T}^{\alpha+T} f(x)\mathrm{dx}$

Some Standard Results:

$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$, $\int_0^\infty \frac{\cos x}{x}\,dx = \infty$, $\int_0^\infty e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{2a}$,

$\int_{-\infty}^\infty e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{a}$, $\int_{-\infty}^\infty e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$, $\int_0^\infty e^{-ax}\,dx = \frac{1}{a}$,

$\int_0^\infty xe^{-x^2}\,dx = \frac{1}{2}$,

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